GAVSS

SVMMATIO

QVARVMDAM

SERIERUM

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SVMMATIO QVARVMDAM SERIERVM SINGVLARIVM

AVCTORE

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EXHIBITA SOCIETATI
D. XXIV. AVGVST. CIƆIƆCCCVIII.

1.

Inter veritates insigniores, ad quas theoria diuisionis circuli aditum aperuit, locum haud vltimum sibi vindicat summatio in Disquiss. Arithmet. p. 636 proposita, non modo propter elegantiam suam peculiarem, miramque foecunditatem, quam fusius exponendi occasionem posthac dabit alia disquisitio, sed ideo quoque, quod eius demonstratio rigorosa atque completa difficultatibus haud vulgaribus premitur. Quae sane eo minus exspectari debuissent, quum non tam in ipsum theorema cadant, quam potius in aliquam theorematis limitationem, qua neglecta demonstratio statim in promtu est, facillimeque e theoria in opere isto explicata deriuatur. Theorema illic exhibitum est in forma sequente. Supponendo n esse numerum primum, denotandoque indefinite omnia residua quadratica ipsius n inter limites 1 et $n-1$ incl. sita per a, omniaque non residua inter eosdem limites iacentia per b, denique per ω arcum $\frac{360^\circ}{n}$, et per k integrum determinatum quemcunque per n non diuisibilem, erit

I. pro valore ipſius n, qui eſt formae $4m+1$,

$\Sigma \cos ak\omega = -\frac{1}{2} \pm \frac{1}{2}\sqrt{n}$

$\Sigma \cos bk\omega = -\frac{1}{2} \mp \frac{1}{2}\sqrt{n}$, adeoque

$\Sigma \cos ak\omega - \Sigma \cos bk\omega = \pm\sqrt{n}$

$\Sigma \sin ak\omega = 0$

$\Sigma \sin bk\omega = 0$

II. pro valore ipſius n, qui eſt formae $4m+3$,

$\Sigma \cos ak\omega = -\frac{1}{2}$

$\Sigma \cos bk\omega = -\frac{1}{2}$

$\Sigma \sin ak\omega = \pm \frac{1}{2}\sqrt{n}$

$\Sigma \sin bk\omega = \mp \frac{1}{2}\sqrt{n}$

$\Sigma \sin ak\omega - \Sigma \sin bk\omega = \pm\sqrt{n}$

Hae ſummationes l. c. omni rigore demonſtratae ſunt, neque alia difficultas hic remanet niſi in determinatione *ſigni* quantitati radicali praefigendi. Nullo quidem negotio oſtendi poteſt, hoc ſignum eatenus a numero k pendere, quod ſemper pro cunctis valoribus ipſius k, qui ſint reſidua quadratica ipſius n, ſignum *idem* valere debeat, et contra ſignum huic oppoſitum pro omnibus valoribus ipſius k, qui ſint non-reſidua quadratica ipſius n. Hinc totum negotium in valore $k=1$ verſabitur, patetque, quam primum ſignum pro hoc valore valens innotuerit, pro omnibus quoque reliquis valoribus ipſius k ſigna ſtatim in promtu fore. Verum enim vero in hac ipſa quaeſtione, quae primo aſpectu inter faciliores referenda videtur, in difficultates improuiſas incidimus, methodusque, qua ducente ſine impedimentis hucusque progreſſi eramus, auxilium vlterius prorſus denegat.

2.

Haud abs re erit, antequam vlterius progrediamur, quaedam exempla ſummationis noſtrae per calculum numericum euoluiſſe: huic vero quasdam obſeruationes generales praemittere conueniet.

I. Si

I. Si in caſu eo, vbi n eſt numerus primus formae $4m+1$, omnia reſidua quadratica ipſius n inter 1 et $\frac{1}{2}(n-1)$ incl. iacentia indefinite per a' exhibentur, omniaque non-reſidua inter eosdem limites per b', conſtat, omnes $n-a'$ inter ipſos a, omnesque $n-b'$ inter b comprehenſos fore: quamobrem quum omnes a', b', $n-a'$, $n-b'$ manifeſto totum complexum numerorum $1, 2, 3 \ldots n-1$ expleant, omnes a' cum omnibus $n-a'$ iuncti omnes a complectentur, et perinde omnes b' cum omnibus $n-b'$ iuncti omnes b comprehendent. Hinc erit

$$\Sigma \cos a k\omega = \Sigma \cos a' k\omega + \Sigma \cos (n-a') k\omega$$
$$\Sigma \cos b k\omega = \Sigma \cos b' k\omega + \Sigma \cos (n-b') k\omega$$
$$\Sigma \sin a k\omega = \Sigma \sin a' k\omega + \Sigma \sin (n-a') k\omega$$
$$\Sigma \sin b k\omega = \Sigma \sin b' k\omega + \Sigma \sin (n-b') k\omega$$

Iam quum habeatur $\cos(n-a')k\omega = \cos a' k\omega$, $\cos(n-b')k\omega = \cos b' k\omega$, $\sin(n-a')k\omega = -\sin a' k\omega$, $\sin(n-b')k\omega = -\sin b' k\omega$, patet ſponte fieri

$$\Sigma \sin a k\omega = \Sigma \sin a' k\omega - \Sigma \sin a' k\omega = 0$$
$$\Sigma \sin b k\omega = \Sigma \sin b' k\omega - \Sigma \sin b' k\omega = 0$$

Summatio coſinuum vero hanc formam aſſumit

$$\Sigma \cos a k\omega = 2 \Sigma \cos a' k\omega$$
$$\Sigma \cos b k\omega = 2 \Sigma \cos b' k\omega$$

vnde fieri debebit

$$1 + 4 \Sigma \cos a' k\omega = \pm \sqrt{n}$$
$$1 + 4 \Sigma \cos b' k\omega = \mp \sqrt{n}$$
$$2 \Sigma \cos a' k\omega - 2 \Sigma \cos b' k\omega = \pm \sqrt{n}$$

II. In caſu eo, vbi n eſt formae $4m+3$, complementum cuiusuis reſidui a ad n erit non-reſiduum, complementumque cuiusuis b erit reſiduum; quocirca omnes $n-a$ conuenient cum omnibus b, omnesque $n-b$ cum omnibus a. Hinc colligitur

$$\Sigma \cos a k\omega = \Sigma \cos (n-b) k\omega = \Sigma \cos b k\omega$$

 quare

quare quum omnes a et b iuncti omnes numeros $1, 2, 3 \ldots n-1$ expleant, adeoque fiat $\Sigma \cos a k\omega + \Sigma \cos b k\omega = \cos k\omega + \cos 2k\omega + 3k\omega +$ etc. $+ \cos nk\omega = -1$, summationes

$$\Sigma \cos a k\omega = -\tfrac{1}{2}$$
$$\Sigma \cos b k\omega = -\tfrac{1}{2}$$

sponte sunt obuiae. Perinde erit

$$\Sigma \sin a k\omega = \Sigma \sin (n-b) k\omega = -\Sigma \sin b k\omega$$

vnde patet, quomodo summationum

$$2\Sigma \sin a k\omega = \pm \sqrt{n}$$
$$2\Sigma \sin b k\omega = \mp \sqrt{n}$$

altera ab altera pendeat.

3.

Ecce iam computum numericum pro aliquot exemplis:

I. Pro $n=5$ adest valor vnus ipsius a', puta $a'=1$, valorque vnus ipsius b', puta $b'=2$; est autem

$$\cos\omega = +0,3090169944 \qquad \cos 2\omega = -0,8090169944$$

adeoque $1 + 4\cos\omega = +\sqrt{5}$, $1 + 4\cos 2\omega = -\sqrt{5}$.

II. Pro $n=13$ adsunt tres valores ipsius a', puta $1, 3, 4$, totidemque valores ipsius b', puta $2, 5, 6$, vnde computamus

$\cos\omega = +0,8854560257$	$\cos 2\omega = +0,5680647467$
$\cos 3\omega = +0,1205366803$	$\cos 5\omega = -0,7485107482$
$\cos 4\omega = -0,3546048870$	$\cos 6\omega = -0,9709418174$
Summa $= +0,6513878190$	Summa $= -1,1513878189$

Hinc $1 + 4\Sigma\cos a'\omega = +\sqrt{13}$, $1 + 4\Sigma\cos b'\omega = -\sqrt{13}$.

III. Pro $n=17$ habemus quatuor valores ipsius a', puta $1, 2, 4, 8$, totidemque valores ipsius b', puta $3, 5, 6, 7$. Hinc computantur cosinus

cos

$\cos\omega = +0,9324722294$	$\cos 3\omega = +0,4457383558$
$\cos 2\omega = +0,7390089172$	$\cos 5\omega = -0,2736629901$
$\cos 4\omega = +0,0922683595$	$\cos 6\omega = -0,6026346364$
$\cos 8\omega = -0,9829730997$	$\cos 7\omega = -0,8502171357$
Summa $= +0,7807764064$	Summa $= -1,2807764065$

Hinc $1 + 4\Sigma\cos a'\omega = +\sqrt{17}$, $1 + 4\Sigma\cos b'\omega = -\sqrt{17}$.

IV. Pro $n = 3$ adeſt valor vnicus ipſius a, puta $a = 1$, cui reſpondet

$\sin\omega = +0,8660254038$

Hinc $2\sin\omega = +\sqrt{3}$.

V. Pro $n = 7$ adſunt valores tres ipſius a, puta $1, 2, 4$: hinc habentur ſinus

$\sin\omega = +0,7818314825$

$\sin 2\omega = +0,9749279122$

$\sin 4\omega = -0,4338837391$

Summa $= +1,3228756556$, adeoque $2\Sigma\sin a\omega = +\sqrt{7}$.

VI. Pro $n = 11$ valores ipſius a ſunt $1, 3, 4, 5, 9$, quibus reſpondent ſinus

$\sin\omega = +0,5406408175$

$\sin 3\omega = +0,9898214419$

$\sin 4\omega = +0,7557495744$

$\sin 5\omega = +0,2817325568$

$\sin 9\omega = -0,9096319954$

Summa $= +1,6583123952$, et proin $2\Sigma\sin a\omega = +\sqrt{11}$.

VII. Pro $n = 19$ valores ipſius a ſunt $1, 4, 5, 6, 7, 9, 11, 16, 17$, quibus reſpondent ſinus

ſin

$$\begin{aligned}
\text{fin}\,\omega &= +\,0{,}3246994692\\
\text{fin}\,4\omega &= +\,0{,}9694002659\\
\text{fin}\,5\omega &= +\,0{,}9965844930\\
\text{fin}\,6\omega &= +\,0{,}9157733267\\
\text{fin}\,7\omega &= +\,0{,}7357239107\\
\text{fin}\,9\omega &= +\,0{,}1645945903\\
\text{fin}\,11\omega &= -\,0{,}4759473930\\
\text{fin}\,16\omega &= -\,0{,}8371664783\\
\text{fin}\,17\omega &= -\,0{,}6142127127
\end{aligned}$$

Summa $= +\,2{,}1794494718$, adeoque $2\Sigma\,\text{fin}\,a\omega = +\sqrt{19}$.

4.

In omnibus hisce exemplis quantitas radicalis fignum pofitivum obtinet, idemque facile pro valoribus maioribus $n=23$, $n=29$ etc. confirmatur, vnde fortis iam probabilitas oritur, hoc generaliter perinde fe habere. Sed demonftratio huius phaenomeni e principiis l. c. expofitis peti nequit, pleniffimoque iure altioris indaginis aeftimanda eft. Propofitum itaque huius commentationis eo tendit, vt demonftrationem rigorofam huius elegantiffimi theorematis, per plures annos olim variis modis incaffum tentatam, tandemque per confiderationes fingulares fatisque fubtiles feliciter perfectam in medium proferamus, fimulque theorema ipfum falua feu potius aucta elegantia fua ad longe maiorem generalitatem euehamus. Coronidis denique loco nexum mirabilem arctiffimum inter hanc fummationem aliudque theorema arithmeticum grauiffimum docebimus. Speramus, hasce disquifitiones non modo per fe geometris gratas fore, fed methodos quoque, per quas haec omnia efficere licuit, quaeque in aliis quoque occafionibus vtiles effe poterunt, ipforum attentione dignas vifum iri.

5.

5.

Petita est demonstratio nostra e consideratione generis singularis progressionum, quarum termini pendent ab expressionibus talibus

$$\frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})\ldots\ldots(1-x^{m-\mu+1})}{(1-x)\quad(1-xx)\quad(1-x^3)\ldots\ldots(1-x^\mu)}$$

Breuitatis caussa talem fractionem per (m, μ) denotabimus, et primo quasdam obseruationes generales circa huiusmodi functiones praemittemus.

I. Quoties m est integer positiuus minor quam μ, functio (m, μ) manifesto euanescit, numeratore factorem $1-x^0$ implicante. Pro $m=\mu$, factores in numeratore identici erunt ordine inuerso cum factoribus in denominatore, vnde erit $(\mu, \mu)=1$; denique pro casu eo, vbi m est integer positiuus maior quam μ, habentur formulae

$$(\mu+1, \mu)=\frac{1-x^{\mu+1}}{1-x}=(\mu+1, 1)$$

$$(\mu+2, \mu)=\frac{(1-x^{\mu+2})\,(1-x^{\mu+1})}{(1-x)\quad(1-xx)}=(\mu+2, 2)$$

$$(\mu+3, \mu)=\frac{(1-x^{\mu+3})\,(1-x^{\mu+2})\,(1-x^{\mu+1})}{(1-x)\quad(1-xx)\quad(1-x^3)}=(\mu+3, 3)$$

etc. siue generaliter

$$(m, \mu)=(m, m-\mu)$$

II. Porro facile confirmatur, haberi generaliter

$$(m, \mu+1)=(m-1, \mu+1)+x^{m-\mu-1}(m-1, \mu)$$

quamobrem, quum perinde sit

$$(m-1, \mu+1)=(m-2, \mu+1)+x^{m-\mu-2}(m-2, \mu)$$
$$(m-2, \mu+1)=(m-3, \mu+1)+x^{m-\mu-3}(m-3, \mu)$$
$$(m-3, \mu+1)=(m-4, \mu+1)+x^{m-\mu-4}(m-4, \mu)\quad \text{etc.,}$$

quae

quae feries continuari poterit vsque ad

$$(\mu+2, \mu+1) = (\mu+1, \mu+1) + x(\mu+1, \mu)$$
$$= (\mu, \mu) + x(\mu+1, \mu)$$

fiquidem m eft integer pofitiuus maior quam $\mu+1$, erit

$$(m, \mu+1) = (\mu, \mu) + x(\mu+1, \mu) + xx(\mu+2, \mu) + x^3(\mu+3, \mu) + \text{etc.}$$
$$+ x^{m-\mu-1}(m-1, \mu)$$

Hinc patet, fi pro aliquo valore determinato ipfius μ quaevis functio (m, μ) integra fit, exiftente m integro pofitiuo, etiam quamuis functionem $(m, \mu+1)$ integram euadere debere. Quare quum fuppofitio illa pro $\mu = 1$ locum habeat, eadem etiam pro $\mu = 2$ valebit, atque hinc etiam pro $\mu = 3$ etc., i. e. generaliter pro valore quocunque integro pofitiuo ipfius m erit (m, μ) functio integra, fiue productum

$(1-x_m)(1-x_m^{-1})(1-x^{m-2}) \ldots (1-x^{m-\mu+1})$ diuifibile per $(1-x)(1-x^2)(1-x^3) \ldots (1-x^\mu)$

6.

Duas iam progreffiones confiderabimus, quae ambae ad fcopum noftrum ducere poffunt. Progreffio prima haec eft

$$1 - \frac{1-x^m}{1-x} + \frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-xx)} - \frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})}{(1-x)(1-xx)(1-x^3)} + \text{etc.}$$

fiue

$1-(m, 1)+(m, 2)-(m, 3)+(m, 4)-$ etc.

quam breuitatis cauffa per $f(x, m)$ denotabimus. Primo ftatim obuium eft, quoties m fit numerus integer pofitiuus, hanc feriem poft terminum fuum $m+1^{\text{tum}}$ (qui fit $= \pm 1$) *abrumpi*, adeoque in hoc cafu fummam fieri debere functionem finitam integram ipfius x. Porro per art. 5. II. patet, generaliter pro valore quocunque ipfius m haberi

1 = 1

$$1 = 1$$
$$-(m, 1) = -(m-1, 1) - x^{m-1}$$
$$+(m, 2) = +(m-1, 2) + x^{m-2}(m-1, 1)$$
$$-(m, 3) = -(m-1, 3) - x^{m-3}(m-1, 2) \text{ etc.}$$

adeoque

$$f(x, m) = 1 - x^{m-1} - (1 - x^{m-2})(m-1, 1) + (1 - x^{m-3})(m-1, 2)$$
$$- (1 - x^{m-4})(m-1, 3) + \text{ etc.}$$

Sed manifesto fit

$$(1 - x^{m-2})(m-1, 1) = (1 - x^{m-1})(m-2, 1)$$
$$(1 - x^{m-3})(m-1, 2) = (1 - x^{m-1})(m-2, 2)$$
$$(1 - x^{m-4})(m-1, 3) = (1 - x^{m-1})(m-2, 3) \text{ etc.}$$

vnde deducimus aequationem

$$f(x, m) = (1 - x^{m-1}) f(x, m-2) \quad \ldots\ldots\ldots\ldots \quad [1]$$

7.

Quum pro $m = 0$ fiat $f(x, m) = 1$, per formulam modo inventam erit

$$f(x, 2) = 1 - x$$
$$f(x, 4) = (1 - x)(1 - x^3)$$
$$f(x, 6) = (1 - x)(1 - x^3)(1 - x^5)$$
$$f(x, 8) = (1 - x)(1 - x^3)(1 - x^5)(1 - x^7) \text{ etc.}$$

siue generaliter pro valore quocunque pari ipsius m

$$f(x, m) = (1 - x)(1 - x^3)(1 - x^5) \ldots (1 - x^{m-1}) \ldots \quad [2]$$

Contra quum pro $m = 1$ fiat $f(x, m) = 0$, erit etiam

$$f(x, 3) = 0$$
$$f(x, 5) = 0$$
$$f(x, 7) = 0 \text{ etc.}$$

siue generaliter pro valore quocunque impari ipsius m

$$f(x, m) = 0$$

Ceterum summatio posterior iam inde deriuari potuisset, quod in progressione

$1-(m,1)+(m,2)-(m,3)+\text{etc.}+(m,m-1)-(m,m)$

terminus vltimus primum destruit, penultimus secundum etc.

8.

Ad scopum quidem nostrum sufficit casus is, vbi m est integer positiuus impar: sed propter rei singularitatem etiam de casibus iis vbi m vel fractus vel negatiuus est pauca adiecisse haud poenitebit. Manifesto tunc series nostra haud amplius abrumpetur, sed in infinitum excurret, facileque insuper perspicitur, diuergentem eam fieri, quoties ipsi x valor minor quam 1 tribuatur, quapropter ipsius summatio ad valores ipsius x qui sint maiores quam 1 restringi debebit.

Per formulam [1] art. 6. habemus

$$f(x,-2)=\frac{1}{1-\frac{1}{x}}$$

$$f(x,-4)=\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{1-\frac{1}{x^3}}$$

$$f(x,-6)=\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{1-\frac{1}{x^3}}\cdot\frac{1}{1-\frac{1}{x^5}}\quad\text{etc.}$$

ita vt valor functionis $f(x,m)$ etiam pro valore negatiuo integro pari ipsius m in terminis finitis assignabilis sit. Pro reliquis vero valoribus ipsius m functionem $f(x,m)$ in *productum infinitum* sequenti modo conuertemus.

Crescente m in valorem negatiuum *infinitum*, functio $f(x,m)$ transit in

$$1+\frac{1}{x-1}+\frac{1}{x-1}\cdot\frac{1}{xx-1}+\frac{1}{x-1}\cdot\frac{1}{xx-1}\cdot\frac{1}{x^3-1}+\text{etc.}$$

Haec

Haec itaque feries aequalis eft producto infinito

$$\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{1-\frac{1}{x^3}}\cdot\frac{1}{1-\frac{1}{x^5}}\cdot\frac{1}{1-\frac{1}{x^7}}\text{ etc. in infin.}$$

Porro quum generaliter fit

$$f(x,m)=f(x,m-2\lambda).(1-x^{m-1})(1-x^{m-3})(1-x^{m-5})\ldots(1-x^{m-2\lambda+1})$$

erit

$$f(x,m)=f(x,-\infty).(1-x^{m-1})(1-x^{m-3})(1-x^{m-5})\text{ etc. in infin.}$$

$$=\frac{1-x^{m-1}}{1-x^{-1}}\cdot\frac{1-x^{m-3}}{1-x^{-3}}\cdot\frac{1-x^{m-5}}{1-x^{-5}}\cdot\frac{1-x^{m-7}}{1-x^{-7}}\text{ etc. in infin.}$$

quos factores tandem continuo magis ad vnitatem conuergere palam eft.

Attentionem peculiarem meretur cafus $m=-1$, vbi fit

$$f(x,-1)=1+x^{-1}+x^{-3}+x^{-6}+x^{-10}+\text{ etc.}$$

Haec itaque feries aequatur producto infinito

$$\frac{1-x^{-2}}{1-x^{-1}}\cdot\frac{1-x^{-4}}{1-x^{-3}}\cdot\frac{1-x^{-6}}{1-x^{-5}}\text{ etc.}$$

fiue fcribendo x pro x^{-1}, erit

$$1+x+x^3+x^6+\text{etc.}=\frac{1-xx}{1-x}\cdot\frac{1-x^4}{1-x^3}\cdot\frac{1-x^6}{1-x^5}\cdot\frac{1-x^8}{1-x^7}\text{ etc.}$$

Haec aequalitas inter duas expreffiones abftrufiores, ad quas alia occafione reueniemus, valde fane eft memorabilis.

9.

Secundo loco confiderabimus progreffionem hancce

$$1+x^{\frac{1}{2}}\frac{1-x^m}{1-x}+x\frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-xx)}+x^{\frac{3}{2}}\frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})}{(1-x)(1-xx)(1-x^3)}$$

etc., fiue

$$1+x^{\frac{1}{2}}(m,1)+x(m,2)+x^{\frac{3}{2}}(m,3)+xx(m,4)\text{ etc.}$$

quam per $F(x,m)$ denotabimus. Reftringemus hanc disquifitionem ad cafum eum, vbi m eft integer pofitiuus, ita vt haec quoque

 feries

feries femper abrumpatur cum termino $m+1^{to}$, qui eft $=x^{\frac{1}{2}m}(m,m)$. Quum fit $(m,m)=1$, $(m,m-1)=(m,1)$, $(m,m-2)=(m,2)$ etc., progreffio ita quoque exhiberi poterit:

$$F(x,m)=x^{\frac{1}{2}m}+x^{\frac{1}{2}m-1}(m,1)+x^{\frac{1}{2}m-2}(m,2)+x^{\frac{1}{2}m-3}(m,3)+\text{etc.}$$

Hinc fit

$$(1+x^{\frac{1}{2}m+\frac{1}{2}})F(x,m)=1+x^{\frac{1}{2}}(m,1)+x(m,2)+x^{\frac{3}{2}}(m,3)+\text{etc.}$$
$$+x^{\frac{1}{2}}.x^{m}+x.x^{m-1}(m+1)+x^{\frac{3}{2}}.x^{m-2}(m+2)+\text{etc.}$$

Quare quum habeatur (art. 5. II.)

$(m,1)+x^{m}=(m+1,1)$

$(m,2)+x^{m-1}(m,1)=(m+1,2)$

$(m,3)+x^{m-2}(m,2)=(m+1,3)$ etc.,

prouenit

$$(1+x^{\frac{1}{2}m+\frac{1}{2}})\,F(x,m)=F(x,m+1)\quad\ldots\ldots\ldots\quad[3]$$

Sed fit $F(x,0)=1$: quamobrem erit

$F(x,1)=1+x^{\frac{1}{2}}$

$F(x,2)=(1+x^{\frac{1}{2}})(1+x)$

$F(x,3)=(1+x^{\frac{1}{2}})(1+x)(1+x^{\frac{3}{2}})$ etc.,

fiue generaliter

$$F(x,m)=(1+x^{\frac{1}{2}})(1+x)(1+x^{\frac{3}{2}})\ldots\ldots(1+x^{\frac{1}{2}m})\quad\ldots\ldots\quad[4]$$

10.

Praemiffis hisce disquifitionibus praeliminaribus iam propius ad propofitum noftrum accedamus. Quum pro valore primo ipfius n quadrata $1, 4, 9 \ldots (\frac{1}{2}(n-1))^2$ omnia inter fe incongrua fint fecundum modulum n, patet, illorum refidua minima fecundum hunc modulum cum numeris a identica effe debere, adeoque

$$\Sigma\cos ak\omega=\cos k\omega+\cos 4k\omega+\cos 9k\omega+\text{etc.}+\cos(\tfrac{1}{2}(n-1))^2k\omega$$
$$\Sigma\sin ak\omega=\sin k\omega+\sin 4k\omega+\sin 9k\omega+\text{etc.}+\sin(\tfrac{1}{2}(n-1))^2k\omega$$

Perinde

Perinde quum eadem quadrata $1, 4, 9 \ldots (\frac{1}{2}(n-1))^2$ ordine inverfo congrua fint his $(\frac{1}{2}(n+1))^2$, $(\frac{1}{2}(n+3))^2$, $(\frac{1}{2}(n+5))^2 \ldots (n-1)^2$, etiam erit

$$\Sigma \cos ak\omega = \cos(\tfrac{1}{2}(n+1))^2 k\omega + \cos(\tfrac{1}{2}(n+3))^2 k\omega + \text{etc.} + \cos(n-1)^2 k\omega$$

$$\Sigma \operatorname{fin} ak\omega = \operatorname{fin}(\tfrac{1}{2}(n+1))^2 k\omega + \operatorname{fin}(\tfrac{1}{2}(n+3))^2 k\omega + \text{etc.} + \operatorname{fin}(n-1)^2 k\omega$$

Statuendo itaque

$$T = 1 + \cos k\omega + \cos 4k\omega + \cos 9k\omega + \text{etc.} + \cos(n-1)^2 k\omega$$

$$U = \operatorname{fin} k\omega + \operatorname{fin} 4k\omega + \operatorname{fin} 9k\omega + \text{etc.} + \operatorname{fin}(n-1)^2 k\omega$$

erit

$$1 + 2\Sigma \cos ak\omega = T$$

$$2\Sigma \operatorname{fin} ak\omega = U$$

Hinc patet, fummationes, quales in art. 1. propofitae funt, pendere a fummatione ferierum T et U, quocirca, miffis illis, disquifitionem noftram his adaptabimus, eaque generalitate abfoluemus, vt non modo valores primos ipfius n, fed quoscunque compofitos complectatur. Numerum k autem fupponemus ad n primum effe: nullo enim negotio cafus is, vbi k et n diuiforem communem haberent, ad hunc reduci poterit.

11.

Defignemus quantitatem imaginariam $\sqrt{-1}$ per i, ftatuamusque

$$\cos k\omega + i \operatorname{fin} k\omega = r$$

vnde erit $r^n = 1$, fiue r radix aequationis $r^n - 1 = 0$. Facile perfpicietur, omnes numeros $k, 2k, 3k \ldots (n-1)k$ per n non diuifibiles atque inter fe fecundum modulum n incongruos effe: hinc poteftates ipfius r

$$1, r, rr, r^3 \ldots r^{n-1}$$

omnes erunt inaequales, fingulae vero quoque aequationi $x^n - 1 = 0$ fatisfacient. Hanc ob cauffam hae poteftates *omnes* radices aequationis $x^n - 1 = 0$ repraefentabunt.

Hae

Hae concluſiones non valerent, ſi k diuiſorem communem haberet cum n. Si enim ν eſſet talis diuiſor communis, foret $k . \frac{n}{\nu}$ per n diuiſibilis, adeoque poteſtas inferior quam r^n, puta $r^{\frac{n}{\nu}}$, vnitati aequalis. In hoc itaque caſu poteſtates ipſius r ad ſummum $\frac{n}{\nu}$ radices aequationis $x^n - 1 = 0$ exhibebunt, et quidem revera tot radices diuerſas ſiſtent, ſi ν eſt diuiſor communis *maximus* numerorum k, n. In caſu noſtro, vbi k et n ſupponuntur inter ſe primi, r commode dici poteſt *radix propria* aequationis $x^n - 1 = 0$: contra in caſu altero, vbi k et n haberent diuiſorem communem (maximum) ν, r vocaretur *radix impropria* illius aequationis, manifeſto autem tunc eadem r foret radix propria aequationis $x^{\frac{n}{\nu}} - 1 = 0$. Radix impropria ſimpliciſſima eſt vnitas, in eoque caſu, vbi n eſt numerus primus, impropriae aliae omnino non dabuntur.

12.

Quodſi iam ſtatuimus

$$W = 1 + r + r^4 + r^9 + \text{etc.} + r^{(n-1)^2},$$

patet fieri $W = T + i U$, adeoque T eſſe partem realem ipſius W, atque U prodire ex parte imaginaria ipſius W factore i ſuppreſſo. Totum itaque negotium reducitur ad inuentionem ſummae W: ad hunc finem vel ſeries in art 6. conſiderata, vel ea quam in art. 9. ſummare docuimus, adhiberi poteſt, prior tamen minus idonea eſt in caſu eo vbi n eſt numerus par. Nihilominus lectoribus gratum fore ſperamus, ſi caſum eum vbi n impar eſt ſecundum methodum duplicem tractemus.

Supponamus itaque primo, n eſſe numerum imparem, r deſignare radicem propriam aequationis $x^n - 1 = 0$ quamcunque, et in functione $f(x, m)$ ſtatui $x = r$, atque $m = n - 1$. Hinc patet fieri

$\frac{1 - x^m}{1 - x}$

$$\frac{1-x^m}{1-x} = \frac{1-r^{-1}}{1-r} = -r^{-1}$$

$$\frac{1-x^{m-1}}{1-xx} = \frac{1-r^{-2}}{1-rr} = -r^{-2}$$

$$\frac{1-x^{m-2}}{1-x^3} = \frac{1-r^{-3}}{1-r^3} = -r^{-3} \text{ etc.}$$

vsque ad

$$\frac{1-x}{1-x^m} = \frac{1-r^{-m}}{1-r^m} = -r^{-m}$$

(Haud ſuperfluum erit monere, has aequationes eatenus tantum valere, quatenus r ſupponitur radix propria: ſi enim eſſet r radix impropria, in quibusdam illarum fractionum numerator et denominator ſimul euaneſcerent, adeoque fractiones indeterminatae fierent).

Hinc deducimus aequationem ſequentem

$$f(r, n-1) = 1 + r^{-1} + r^{-3} + r^{-6} + \text{etc.} + r^{-\frac{1}{2}(n-1)n}$$
$$= (1-r)(1-r^3)(1-r^5) \ldots\ldots (1-r^{n-2})$$

Eadem aequatio etiamnum valebit, ſi pro r ſubſtituitur r^λ, deſignante λ integrum quemcunque ad n primum: tunc enim etiam r^λ erit radix propria aequationis $x^n - 1 = 0$. Scribamus itaque pro r, r^{n-2} ſiue quod idem eſt r^{-2}, eritque

$$1 + r^2 + r^6 + r^{12} + \text{etc.} + r^{(n-1)n} = (1-r^{-2})(1-r^{-6})(1-r^{-10}) \ldots (1-r^{-2(n-2)})$$

Multiplicemus vtramque partem huius aequationis per

$$r \,.\, r^3 \,.\, r^5 \ldots\ldots r^{(n-2)} = r^{\frac{1}{4}(n-1)^2}$$

prodibitque, propter

$$r^{2+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n-3)^2}, \quad r^{(n-1)n+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n+1)^2}$$
$$r^{6+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n-5)^2}, \quad r^{(n-2)(n-1)+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n+3)^2}$$
$$r^{12+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n-7)^2}, \quad r^{(n-3)(n-2)+\frac{1}{4}(n-1)^2} = r^{\frac{1}{4}(n+5)^2} \text{ etc.}$$

aequatio

aequatio sequens

$$r^{\frac{1}{4}(n-1)^2} + r^{\frac{1}{4}(n-3)^2} + r^{\frac{1}{4}(n-5)^2} + \text{etc.} + r + 1$$
$$+ r^{\frac{1}{4}(n+1)^2} + r^{\frac{1}{4}(n+3)^2} + r^{\frac{1}{4}(n+5)^2} + \text{etc.} + r^{\frac{1}{4}(2n-1)^2}$$
$$= (r - r^{-1})(r^3 - r^{-3})(r^5 - r^{-5}) \ldots (r^{n-2} - r^{-n+2})$$

aut, partibus membri primi aliter dispositis,

$$1 + r + r^4 + \text{etc.} + r^{(n-1)^2} = (r - r^{-1})(r^3 - r^{-3}) \ldots (r^{n-2} - r^{-n+2})$$
$$\ldots\ldots [5]$$

13.

Factores membri secundi aequationis [5] ita quoque exhiberi possunt

$$r - r^{-1} = -(r^{n-1} - r^{-n+1})$$
$$r^3 - r^{-3} = -(r^{n-3} - r^{-n+3})$$
$$r^5 - r^{-5} = -(r^{n-5} - r^{-n+5}) \text{ etc.}$$

vsque ad

$$r^{n-2} - r^{-n+2} = -(r^2 - r^{-2})$$

quo pacto aequatio ista hanc formam assumit:

$$W = (-1)^{\frac{1}{2}(n-1)}(r^2 - r^{-2})(r^4 - r^{-4})(r^6 - r^{-6}) \ldots (r^{n-1} - r^{-n+1})$$

Multiplicando hanc aequationem per [5] in forma primitiua, prodit

$$W^2 = (-1)^{\frac{1}{2}(n-1)}(r - r^{-1})(r^2 - r^{-2})(r^3 - r^{-3}) \ldots (r^{n-1} - r^{-n+1})$$

vbi $(-1)^{\frac{1}{2}(n-1)}$ est vel $= +1$ vel $= -1$, prout n est formae $4\mu + 1$, vel formae $4\mu + 3$. Hinc

$$W^2 = \pm r^{\frac{1}{2}n(n-1)}(1 - r^{-2})(1 - r^{-4})(1 - r^{-6}) \ldots (1 - r^{-2(n-1)})$$

Sed nullo negotio perspicitur, r^{-2}, r^{-4}, $r^{-6} \ldots r^{-2n+2}$ exhibere omnes radices aequationis $x^n - 1 = 0$, radice $x = 1$ excepta, vnde locum habere debebit aequatio identica indefinita

$$(x - r^{-2})(x - r^{-4})(x - r^{-6}) \ldots (x - r^{-2n+2})$$
$$= x^{n-1} + x^{n-2} + x^{n-3} + \text{etc.} + x + 1$$

Quamobrem statuendo $x = 1$, fiet

$$(1 - r^{-2})(1 - r^{-4})(1 - r^{-6}) \ldots (1 - r^{-2n+2}) = n$$

quo

et quum manifesto sit $r^{\frac{1}{2}n(n-1)} = 1$, aequatio nostra transit in hanc

$W^2 = \pm n$ . . . . [6]

In casu itaque eo, vbi n est formae $4\mu + 1$, fiet

$W = \pm \sqrt{n}$, et proin $T = \pm \sqrt{n}$, $U = 0$.

Contra in casu altero, vbi n est formae $4\mu + 3$, fiet

$W = \pm i \sqrt{n}$, adeoque $T = 0$, $U = \pm \sqrt{n}$.

14.

Methodus art. praec. valorem tantummodo absolutum aggregatorum T, U assignat, ambiguumque linquit, vtrum statuere oporteat T in casu priori atque U in casu posteriori $= + \sqrt{n}$, an $= - \sqrt{n}$. Hoc autem, saltem pro casu eo vbi $k = 1$, ex aequatione 5 sequenti modo decidere licebit. Quum sit, pro $k = 1$,

$r - r^{-1} = 2i \sin \omega$

$r^3 - r^{-3} = 2i \sin 3\omega$

$r^5 - r^{-5} = 2i \sin 5\omega$ etc.,

aequatio ista transmutatur in

$W = (2i)^{\frac{1}{2}(n-1)} \sin \omega \sin 3\omega \sin 5\omega \ldots \sin (n-2)\omega$

Iam in casu eo, vbi n est formae $4\mu + 1$, in serie numerorum imparium

$1, 3, 5, 7 \ldots \frac{1}{2}(n-3), \frac{1}{2}(n+1) \ldots (n-2)$

reperiuntur $\frac{1}{4}(n-1)$, qui sunt minores quam $\frac{1}{2}n$, hisque manifesto respondent sinus positiui; contra reliqui $\frac{1}{4}(n-1)$ erunt maiores quam $\frac{1}{2}n$, hisque sinus negatiui respondebunt: quapropter productum omnium sinuum statuendum est aequale producto e quantitate positiua in multiplicatorem $(-1)^{\frac{1}{4}(n-1)}$, adeoque W aequalis erit producto e quantitate reali positiua in i^{n-1} siue in 1, quoniam $i^4 = 1$, atque $n - 1$ per 4 diuisibilis: i. e. quantitas W erit realis positiua, vnde necessario esse debebit

$W = + \sqrt{n}$, $T = + \sqrt{n}$.

In casu altero, vbi n est formae $4\mu+3$ in serie numerorum imparium

$1, 3, 5, 7 \ldots\ldots \frac{1}{2}(n-1), \frac{1}{2}(n+3) \ldots\ldots (n-2)$

priores $\frac{1}{4}(n+1)$ erunt minores quam $\frac{1}{2}n$; reliqui $\frac{1}{4}(n-3)$ autem maiores. Hinc inter sinus arcuum ω, 3ω, $5\omega \ldots\ldots (n-2)\omega$ negatiui erunt $\frac{1}{4}(n-3)$, adeoque W erit productum ex $i^{\frac{1}{2}(n-1)}$ in quantitatem realem positiuam in $(-1)^{\frac{1}{4}(n-3)}$; factor tertius est $=i^{\frac{1}{2}(n-3)}$, qui cum primo iunctus producit $i^{n-2}=i$, quoniam $i^{n-3}=1$. Quamobrem necessario erit

$W=+i\sqrt{n}$, atque $U=+\sqrt{n}$.

15.

Iam ostendemus, quo pacto eaedem conclusiones e progressione in art. 9. considerata deduci possint. Scribamus in aequ. [4] pro $x^{\frac{1}{2}}$, $-y^{-1}$, eritque

$$1-y^{-1}\frac{1-y^{-2m}}{1-y^{-2}}+y^{-2}\frac{(1-y^{-2m})(1-y^{-2m+2})}{(1-y^{-2})(1-y^{-4})}$$
$$-y^{-3}\frac{(1-y^{-2m})(1-y^{-2m+2})(1-y^{-2m+4})}{(1-y^{-2})(1-y^{-4})(1-y^{-6})}+\text{etc.}$$

vsque ad terminum $m+1^{\text{tum}}$

$$=(1-y^{-1})(1+y^{-2})(1-y^{-3})(1+y^{-4})\ldots(1\pm y^{-m})\ldots [7]$$

Quodsi hic pro y accipitur radix propria aequationis $y^n-1=0$, puta r, atque simul statuitur $m=n-1$, erit

$$\frac{1-y^{-2m}}{1-y^{-2}}=\frac{1-r^2}{1-r^{-2}}=-r^2$$
$$\frac{1-y^{-2m+2}}{1-y^{-4}}=\frac{1-r^4}{1-r^{-4}}=-r^4$$
$$\frac{1-y^{-2m+4}}{1-y^{-6}}=\frac{1-r^6}{1-r^{-6}}=-r^6 \text{ etc.}$$

vsque ad

$$\frac{1-y^{-2}}{1-y^{-2m}}=\frac{1-r^{2n-2}}{1-r^{-2n+2}}=-r^{2n-2}$$

vbi

vbi notandum, nullum denominatorum $1-r^{-2}$, $1-r^{-4}$ etc. fieri $=0$. Hinc aequatio [7] hancce formam assumit

$$1+r+r^4+r^9+\text{etc.}+r^{(n-1)^2}=$$
$$(1-r^{-1})(1+r^{-2})(1-r^{-3})\ldots\ldots(1+r^{-n+1})$$

Multiplicando in membro secundo huius aequationis terminum primum per vltimum, secundum per penultimum etc., habemus

$$(1-r^{-1})(1+r^{-n+1})=r-r^{-1}$$
$$(1+r^{-2})(1-r^{-n+2})=r^{n-2}-r^{-n+2}$$
$$(1-r^{-3})(1+r^{-n+3})=r^3-r^{-3}$$
$$(1+r^{-4})(1-r^{-n+4})=r^{n-4}-r^{-n+4}\ \text{etc.}$$

Ex his productis partialibus facile perspicietur conflari productum

$$(r-r^{-1})(r^3-r^{-3})(r^5-r^{-5})\ldots\ldots(r^{n-4}-r^{-n+4})(r^{n-2}-r^{-n+2})$$

quod itaque erit $=1+r+r^4+r^9+\text{etc.}+r^{(n-1)^2}=W$. Haec aequatio identica est cum aequ. [5] in art. 12. e progressione prima deriuata, ratiociniaque dein reliqua eodem modo adstruentur, vt in artt. 13. et 14.

16.

Transimus ad casum alterum, vbi n est numerus par. Sit primo n formae $4\mu+2$ siue impariter par, patetque, numeros $\frac{1}{4}nn$, $(\frac{1}{2}n+1)^2-1$, $(\frac{1}{2}n+2)^2-4$ etc. siue generaliter $(\frac{1}{2}n+\lambda)^2-\lambda\lambda$ per $\frac{1}{2}n$ diuisos producere quotientes impares, adeoque secundum modulum n congruos fieri ipsi $\frac{1}{2}n$. Hinc colligitur, si r sit radix propria aequationis $x^n-1=0$, adeoque $r^{\frac{1}{2}n}=-1$, fieri

$$r^{(\frac{1}{2}n)^2}=-1$$
$$r^{(\frac{1}{2}n+1)^2}=-r$$
$$r^{(\frac{1}{2}n+2)^2}=-r^4$$
$$r^{(\frac{1}{2}n+3)^2}=-r^9\ \text{etc.}$$

Hinc

Hinc in progreſſione

$$1+r+r^4+r^9+\text{etc.}+r^{(n-1)^2}$$

terminus $r^{(\frac{1}{2}n)^2}$ deſtruet primum, ſequens ſecundum etc., adeoque erit $W=0$, $T=0$, $U=0$.

17.

Supereſt caſus, vbi n eſt formae 4μ ſiue pariter par. Hic generaliter $(\frac{1}{2}n+\lambda)^2-\lambda\lambda$ diuiſibilis erit per n, adeoque

$$r^{(\frac{1}{2}n\pm\lambda)^2}=r^{\lambda\lambda}$$

Hinc in ſerie

$$1+r+r^4+r^9+\text{etc.}+r^{(n-1)^2}$$

terminus $r^{(\frac{1}{2}n)^2}$ aequalis erit primo, ſequens ſecundo etc., ita vt fiat

$$W=2\,(1+r+r^4+r^9+\text{etc.}+r^{(\frac{1}{2}n-1)^2})$$

Iam ſupponamus, in aequ. 7. art. 15. ſtatui $m=\frac{1}{2}n-1$, et pro y accipi radicem propriam aequationis $y^n-1=0$, puta r. Tunc perinde vt in art. 15. aequatio ſequentem formam obtinet:

$$1+r+r^4+\text{etc.}+r^{(\frac{1}{2}n-1)^2}$$
$$=(1-r^{-1})\,(1+r^{-2})\,(1-r^{-3})\ldots\ldots(1-r^{-\frac{1}{2}n+1})$$

ſiue

$$W=2(1-r^{-1})(1+r^{-2})(1-r^{-3})(1+r^{-4})\ldots(1-r^{-\frac{1}{2}n+1})\;\ldots\;[8]$$

Porro quum ſit $r^{\frac{1}{2}n}=-1$, adeoque

$$1+r^{-2}=-r^{\frac{1}{2}n-2}\,(1-r^{-\frac{1}{2}n+2})$$
$$1+r^{-4}=-r^{\frac{1}{2}n-4}\,(1-r^{-\frac{1}{2}n+4})$$
$$1+r^{-6}=-r^{\frac{1}{2}n-6}\,(1-r^{-\frac{1}{2}n+6})\ \text{etc.}$$

productum-

productumque e factoribus $-r^{\frac{1}{2}n-2}$, $-r^{\frac{1}{2}n-4}$, $-r^{\frac{1}{2}n-6}$ etc. vsque ad $-r^2$ fiat $=(-1)^{\frac{1}{4}n-1}\, r^{\frac{1}{16}nn-\frac{1}{4}n}$, aequatio praecedens ita quoque exhiberi potest

$$W=2\,(-1)^{\frac{1}{4}n-1}\, r^{\frac{1}{16}nn-\frac{1}{4}n}\,(1-r^{-1})\,(1-r^{-2})\,(1-r^{-3})$$
$$(1-r^{-4})\ldots\ldots(1-r^{-\frac{1}{2}n+1})$$

Quum habeatur

$$1-r^{-1}=-r^{-1}\,(1-r^{-n+1})$$
$$1-r^{-2}=-r^{-2}\,(1-r^{-n+2})$$
$$1-r^{-3}=-r^{-3}\,(1-r^{-n+3})\ \text{etc.}$$

erit

$$(1-r^{-1})\,(1-r^{-2})\,(1-r^{-3})\ldots(1-r^{-\frac{1}{2}n+1})$$
$$=(-1)^{\frac{1}{2}n-1}\,r^{-\frac{1}{8}nn+\frac{1}{4}n}(1-r^{-\frac{1}{2}n-1})(1-r^{-\frac{1}{2}n-2})(1-r^{-\frac{1}{2}n-3})\ldots(1-r^{-n+1})$$

adeoque

$$W=2\,(-1)^{\frac{3}{4}n-2}\,r^{-\frac{1}{16}nn}\,(1-r^{-\frac{1}{2}n-1})\,(1-r^{-\frac{1}{2}n-2})$$
$$(1-r^{-\frac{1}{2}n-3})\ldots\ldots(1-r^{-n+1})$$

Multiplicando hunc valorem ipsius W per prius inuentum, adiungendoque vtrimque factorem $1-r^{-\frac{1}{2}n}$, prodit

$$(1-r^{-\frac{1}{2}n})\,W^2=4\,(-1)^{n-3}\,r^{-\frac{1}{4}n}\,(1-r^{-1})\,(1-r^{-2})$$
$$(1-r^{-3})\ldots\ldots(1-r^{-n+1})$$

Sed fit

$$1-r^{-\frac{1}{2}n}=2$$
$$(-1)^{n-3}=-1$$
$$r^{-\frac{1}{4}n}=-r^{\frac{1}{4}n}$$
$$(1-r^{-1})\,(1-r^{-2})\,(1-r^{-3})\ldots\ldots(1-r^{-n+1})=n$$

Vnde tandem concluditur

$$W^2=2\,r^{\frac{1}{4}n}n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad [9]$$

Iam facile perspicietur, $r^{\frac{1}{4}n}$ esse vel $=+i$ vel $=-i$, prout scilicet k vel formae $4\mu+1$ sit, vel formae $4\mu+3$. Et quum sit $2i=(1+i)^2$, $-2i=(1-i)^2$, erit

in

in casu eo, vbi k est formae $4\mu+1$,

$W = \pm(1+i)\sqrt{n}$, adeoque $T = U = \pm\sqrt{n}$

in casu altero autem, vbi k est formae $4\mu+3$,

$W = \pm(1-i)\sqrt{n}$, adeoque $T = -U = \pm\sqrt{n}$.

18.

Methodus art. praec. valores absolutos functionum T, U suppeditauit, conditionesque assignauit, sub quibus signa aequalia vel opposita illis tribuenda sint: sed signa ipsa hinc nondum determinantur. Hoc pro eo casu, vbi statuitur $k=1$, sequenti modo supplebimus.

Statuamus $\varrho = \cos\frac{1}{n}\omega + i\sin\frac{1}{n}\omega$, ita vt fiat $r=\varrho\varrho$, patetque, propter $\varrho^n = -1$ aequationem 8 ita exhiberi posse

$W = (1+\varrho^{n-2})(1+\varrho^{-4})(1+\varrho^{n-6})(1+\varrho^{-8})\ldots(1+\varrho^{-n+4})(1+\varrho^{2})$

siue factoribus alio ordine dispositis

$W = (1+\varrho^{2})(1+\varrho^{-4})(1+\varrho^{6})(1+\varrho^{-8})\ldots(1+\varrho^{-n+4})(1+\varrho^{n-2})$

Iam fit

$$1+\varrho^{2} = 2\varrho\cos\tfrac{1}{2}\omega$$
$$1+\varrho^{-4} = 2\varrho^{-2}\cos\omega$$
$$1+\varrho^{6} = 2\varrho^{3}\cos\tfrac{3}{2}\omega$$
$$1+\varrho^{-8} = 2\varrho^{-4}\cos 2\omega \text{ etc.}$$

vsque ad

$$1+\varrho^{-n+4} = 2\varrho^{-\frac{1}{2}n+2}\cos(\tfrac{1}{4}n-1)\omega$$
$$1+\varrho^{n-2} = 2\varrho^{\frac{1}{2}n-1}\cos(\tfrac{1}{4}n-\tfrac{1}{2})\omega$$

Quamobrem habetur

$$W = 2^{\frac{1}{2}n-1}\varrho^{\frac{1}{4}n}\cos\tfrac{1}{2}\omega\cos\omega\cos\tfrac{3}{2}\omega\ldots\cos(\tfrac{1}{4}n-\tfrac{1}{2})\omega$$

Cosinus in hoc productum ingredientes manifesto omnes positiui sunt, factor $\varrho^{\frac{1}{4}n}$ autem fit $= \cos 45° + i\sin 45° = (1+i)\sqrt{\tfrac{1}{2}}$. Hinc colligimus, W esse productum ex $1+i$ in quantitatem realem posituam, vnde necessario esse debebit

$W = (1+i)\sqrt{n}$, $T = +\sqrt{n}$, $U = +\sqrt{n}$.

19.

19.

Operae pretium erit, omnes fummationes haftenus euolutas hic in vnum confpectum colligere. Generaliter fcilicet eft

$T=$	$U=$	prout n eft formae
$\pm\sqrt{n}$	$\pm\sqrt{n}$	4μ
$\pm\sqrt{n}$	o	$4\mu+1$
o	o	$4\mu+2$
o	$\pm\sqrt{n}$	$4\mu+3$

et in cafu eo, vbi k fupponitur $=1$, quantitati radicali fignum pofitiuum tribui debet. Omni itaque iam rigore ea, quae pro valoribus primis ipfius n in art. 3. per inductionem animaduerteramus, demonftrata funt, nihilque fupereft, nifi vt figna pro valoribus quibuscunque ipfius k in omnibus cafibus determinare doceamus. Sed antequam hoc negotium in omni generalitate aggredi liceat, primo cafus eos, vbi n eft numerus primus vel numeri primi poteftas, propius confiderare oportebit.

20.

Sit primo n numerus primus impar, patetque per ea, quae in art. 10. expofuimus, effe $W=1+2\Sigma r^a=1+2\Sigma R^{ak}$, fi ftatuatur $R=\cos\omega+i\sin\omega$, denotante a vt illic indefinite omnia refidua quadratica ipfius n inter 1 et $n-1$ contenta. Quodfi quoque per b indefinite omnia non-refidua quadratica inter eosdem limites exprimimus, nullo negotio perfpicitur, omnes numeros ak congruos fieri fecundum modulum n vel omnibus a vel omnibus b (nullo ordinis refpectu habito), prout k vel refiduum fit vel non-refiduum. Quamobrem in cafu priori erit

$$W=1+2\Sigma R^a=1+R+R^4+R^9+\text{etc.}+R^{(n-1)^2}$$

adeoque $W=+\sqrt{n}$, fi n eft formae $4\mu+1$, atque $W=+i\sqrt{n}$, fi n eft formae $4\mu+3$.

Contra

Contra in caſu altero, vbi k eſt non-reſiduum ipſius n, erit

$$W = 1 + 2\Sigma R^b$$

Hinc quum manifeſto omnes a, b complexum integrum numerorum 1, 2, 3 ... expleant, adeoque ſit

$$\Sigma R^a + \Sigma R^b = R + R^2 + R^3 + \text{etc.} + R^{n-1} = -1$$

fiet

$$W = -1 - 2R^a = -(1 + R + R^4 + R^9 + \text{etc.} + R^{(n-1)^2})$$

adeoque $W = -\sqrt{n}$, ſi n eſt formae $4\mu + 1$, atque $W = -i\sqrt{n}$, ſi n eſt formae $4\mu + 3$.

Hinc itaque colligitur

primo, ſi n eſt formae $4\mu + 1$, atque k reſiduum quadraticum ipſius n,

$$T = +\sqrt{n}, \quad U = 0.$$

ſecundo, ſi n eſt formae $4\mu + 1$, atque k non-reſiduum ipſius n,

$$T = -\sqrt{n}, \quad U = 0.$$

tertio, ſi n eſt formae $4\mu + 3$, atque k reſiduum ipſius n,

$$T = 0, \quad U = +\sqrt{n}.$$

quarto, ſi n eſt formae $4\mu + 3$, atque k non-reſiduum ipſius n,

$$T = 0, \quad U = -\sqrt{n}.$$

21.

Sit ſecundo n quadratum altiorue poteſtas numeri primi imparis p, ſtatuaturque $n = p^{2\kappa} q$, ita vt ſit q vel $= 1$ vel $= p$. Hic ante omnia obſeruare conuenit, ſi λ ſit integer quicunque per p^κ non diuiſibilis, fieri

$$r^{\lambda\lambda} + r^{(\lambda \dotplus p^\kappa q)^2} + r^{(\lambda \dotplus 2p^\kappa q)^2} + r^{(\lambda \dotplus 3p^\kappa q)^2} + \text{etc.} + r^{(\lambda \dotplus n - p^\kappa q)^2}$$

$$= r^{\lambda\lambda} \left\{1 + r^{2\lambda p^\kappa q} + r^{4\lambda p^\kappa q} + r^{6\lambda p^\kappa q} + \text{etc.} + r^{2\lambda(n - p^\kappa q)}\right\}$$

$$= \frac{r^{\lambda\lambda}(1 - r^{2\lambda n})}{1 - r^{2\lambda p^\kappa q}} = 0.$$

Hinc

Hinc facile perſpicietur, fieri

$$W = 1 + r^{p^{2\varkappa}} + r^{4p^{2\varkappa}} + r^{9p^{2\varkappa}} + \text{etc.} + r^{(n-p^\varkappa)^2}$$

Termini enim reliqui progreſſionis

$$1 + r + r^4 + r^9 + \text{etc.} + r^{(n-1)^2}$$

distribui poterunt in $p^\varkappa - 1$ progreſſiones partiales, quae ſingulae ſint $p^\varkappa q$ terminorum, et per transformationem modo traditam ſummas euaneſcentes conficiant.

Hinc colligitur, in caſu eo, vbi fit $q = 1$, ſiue vbi n eſt poteſtas numeri primi cum exponente pari, fieri

$W = p^\varkappa = +\sqrt{n}$, adeoque $T = +\sqrt{n}$, $U = 0$.

Contra in caſu eo, vbi $q = p$, ſiue vbi n eſt poteſtas numeri primi cum exponente impari, ſtatuemus $r^{p^{2\varkappa}} = \varrho$, vnde ϱ erit radix propria aequationis $x^p - 1 = 0$, et quidem $\varrho = \cos \frac{k}{p} 360^\circ + i \sin \frac{k}{p} 360^\circ$, ac dein

$$W = 1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(p^{\varkappa+1}-1)^2}$$
$$= p^\varkappa (1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(p-1)^2})$$

Sed ſumma ſeriei $1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(p-1)^2}$ per art. praec. determinatur, vnde ſponte concluditur, fieri

$W = \pm\sqrt{n} = T$, ſi fuerit p formae $4\mu + 1$,

$W = \pm i\sqrt{n} = iU$, ſi fuerit p formae $4\mu + 3$,

ſigno poſitiuo vel negatiuo valente, prout k fuerit reſiduum vel non-reſiduum ipſius p.

22.

Facile quoque ex iis, quae in artt. 20. et 21. expoſita ſunt, deriuatur propoſitio ſequens, quae infra vſum notabilem nobis praeſtabit. Statuatur

$$W' = 1 + r^h + r^{4h} + r^{9h} + \text{etc.} + r^{h(n-1)^2},$$

denotante h integrum quemcunque per p non diuifibilem, eritque in cafu eo, vbi $n=p$, vel vbi n eft poteftas ipfius p cum exponente impari,

$W' = W$, fi fuerit h refiduum quadraticum ipfius p,

$W' = -W$, fi fuerit h non-refiduum quadraticum ipfius p.

Patet enim, W' oriri ex W, fi pro k fubftituatur kh; in cafu priori autem k et kh fimiles erunt, in pofteriori diffimiles, quatenus funt refidua vel non-refidua ipfius p.

In cafu eo autem, vbi n eft poteftas ipfius p cum exponente pari, manifefto fit $W' = +\sqrt{n}$, adeoque femper $W' = W$.

23.

In artt. 20. 21. 22. confiderauimus numeros primos impares, taliumque poteftates: fupereft itaque cafus, vbi n eft poteftas binarii.

Pro $n=2$ manifefto fit $W = 1 + r = 0$.

Pro $n=4$ prodit $W = 1 + r + r^4 + r^9 = 2 + 2r$: hinc $W = 2 + 2i$, quoties k eft formae $4\mu + 1$, atque $W = 2 - 2i$, quoties k eft formae $4\mu + 3$.

Pro $n=8$ habemus $W = 1 + r + r^4 + r^9 + r^{16} + r^{25} + r^{36} + r^{49}$ $= 2 + 4r + 2r^4 = 4r$. Hinc erit

$W = (1+i)\sqrt{8}$, quoties k eft formae $8\mu + 1$

$W = (-1+i)\sqrt{8}$, quoties k eft formae $8\mu + 3$

$W = (-1-i)\sqrt{8}$, quoties k eft formae $8\mu + 5$

$W = (1-i)\sqrt{8}$, quoties k eft formae $8\mu + 7$.

Si n eft altior poteftas binarii, ftatuamus $n = 2^{2\varkappa}q$, ita vt q fit vel $=1$ vel $=2$, atque $\varkappa$ maior quam 1. Hic ante omnia obfervari debet, fi λ fit integer quicunque per $2^{\varkappa-1}$ non diuifibilis, fieri

$$r^{\lambda\lambda} + r^{(\lambda+2^\varkappa q)^2} + r^{(\lambda+2.2^\varkappa q)^2} + r^{(\lambda+3.2^\varkappa q)^2} + \text{etc.} + r^{(\lambda+n-2^\varkappa q)^2}$$
$$= r^{\lambda\lambda}\{1 + r^{2^{\varkappa+1}\lambda q} + r^{2.2^{\varkappa+1}\lambda q} + r^{3.2^{\varkappa+1}\lambda q} + \text{etc.} + r^{(2n-2^{\varkappa+1}q)\lambda}\}$$
$$= \frac{r^{\lambda\lambda}(1 - r^{2\lambda n})}{1 - r^{2^{\varkappa+1}\lambda q}} = 0.$$

Hinc

Hinc facile perſpicietur, fieri

$$W = 1 + r^{2^{2\kappa-2}} + r^{4\cdot 2^{2\kappa-2}} + r^{9\cdot 2^{2\kappa-2}} + \text{etc.} + r^{(n-2^{\kappa-1})^2}$$

Statuamus $r^{2^{2\kappa-2}} = \varrho$, eritque ϱ radix aequationis $x^{4q} - 1 = 0$, et quidem $\varrho = \cos\frac{k}{4q}360^\circ + i\sin\frac{k}{4q}360^\circ$; dein fiet

$$W = 1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(2^{\kappa+1}q-1)^2}$$
$$= 2^{\kappa-1}(1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(4q-1)^2})$$

Sed ſumma ſeriei $1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(4q-1)^2}$ per ea quae de caſibus $n = 4$, $n = 8$ explicauimus, determinatur, vnde colligimus in caſu eo, vbi $q = 1$, ſiue vbi n eſt poteſtas numeri 4, fieri

$W = (1+i)2^\kappa = (1+i)\sqrt{n}$, ſi fuerit k formae $4\mu + 1$,

$W = (1-i)2^\kappa = (1-i)\sqrt{n}$, ſi fuerit k formae $4\mu + 3$,

quae ſunt ipſiſſimae formulae pro $n = 4$ traditae;

in caſu eo autem, vbi $q = 2$, ſiue vbi n eſt poteſtas binarii cum exponente impari maiori quam 3, fieri

$W = (1+i)2^\kappa\sqrt{2} = (1+i)\sqrt{n}$, ſi fuerit k formae $8\mu + 1$

$W = (-1+i)2^\kappa\sqrt{2} = (-1+i)\sqrt{n}$, ſi fuerit k formae $8\mu + 3$

$W = (-1-i)2^\kappa\sqrt{2} = (-1-i)\sqrt{n}$, ſi fuerit k formae $8\mu + 5$

$W = (1-i)2^\kappa\sqrt{2} = (1-i)\sqrt{n}$, ſi fuerit k formae $8\mu + 7$

quae quoque prorſus conueniunt cum iis, quae pro $n = 8$ tradidimus.

24.

Etiam hic operae pretium erit, rationem ſummae progreſſionis

$$W' = 1 + r^h + r^{4h} + r^{9h} + \text{etc.} + r^{h(n-1)^2}$$

ad W determinare, vbi h integrum quemcunque imparem denotat. Quum W' oriatur ex W, mutando k in kh, valor ipſius W' perinde a forma numeri kh pendebit, vt W a forma ipſius k. Statuamus $\frac{W'}{W} = l$, patetque

I. in

I. in casu eo, vbi $n=4$, vel altior potestas binarii cum exponente pari, fieri

$l=1$, si fuerit h formae $4\mu+1$

$l=-i$, si fuerit h formae $4\mu+3$, atque k formae $4\mu+1$

$l=+i$, si fuerit h formae $4\mu+3$, atque k eiusdem formae;

II. in casu eo, vbi $n=8$, vel altior potestas binarii cum exponente impari, fieri

$l=1$, si fuerit h formae $8\mu+1$,

$l=-1$, si fuerit h formae $8\mu+5$,

$l=+i$, si fuerit vel h formae $8\mu+3$, atque k formae $4\mu+1$,
vel h formae $8\mu+7$, atque k formae $4\mu+3$,

$l=-i$, si fuerit vel h formae $8\mu+3$, atque k formae $4\mu+3$,
vel h formae $8\mu+7$, atque k formae $4\mu+1$.

Per praecc. determinatio summae W pro iis casibus, vbi n est numerus primus vel numeri primi potestas, complete perfecta est: superest itaque, vt eos quoque casus absoluamus, vbi n e pluribus numeris primis compositus est, huc viam nobis sternet theorema sequens.

25.

Theorema. Sit n productum e duobus integris positiuis inter se primis a, b, statuaturque

$$P=1+r^{aa}+r^{4aa}+r^{9aa}+\text{etc.}+r^{(b-1)^2aa}$$

$$Q=1+r^{bb}+r^{4bb}+r^{9bb}+\text{etc.}+r^{(a-1)^2bb}$$

Tum dico fore $W=PQ$.

Demonstr. Designet α indefinite numeros 0, 1, 2, 3 .... $a-1$, β indefinite numeros 0, 1, 2, 3 .... $b-1$, ν indefinite numeros 0, 1, 2, 3 .... $n-1$. Tunc patet esse

$$P=\Sigma r^{aa\beta\beta},\quad Q=\Sigma r^{bb\alpha\alpha},\quad W=\Sigma r^{\nu\nu}$$

Hinc

Hinc erit $PQ = \Sigma r^{aa\beta\beta + bb\alpha\alpha}$, ſubſtituendo pro α et β omnes valores, omnibus modis inter ſe combinatos; hinc porro propter $2ab\alpha\beta = 2\alpha\beta n$, erit $PQ = \Sigma r^{(a\beta + b\alpha)^2}$. Sed nullo negotio perſpicitur, ſingulos valores ipſius $a\beta + b\alpha$ inter ſe diuerſos eſſe, atque alicui valori ipſius ν aequales. Hinc erit $PQ = \Sigma r^{\nu\nu} = W$.

Ceterum notandum eſt, r^{aa} eſſe radicem propriam aequationis $x^b - 1 = 0$, atque r^{bb} radicem propriam aequationis $x^a - 1 = 0$.

26.

Sit porro n productum e tribus numeris inter ſe primis a, b, c, patetque, ſi ſtatuatur $bc = b'$, etiam a et b' inter ſe primos fore; adeoque W productum e duobus factoribus

$$1 + r^{aa} + r^{4aa} + r^{9aa} + \text{etc.} + r^{(b'-1)^2 aa}$$

$$1 + r^{b'b'} + r^{4b'b'} + r^{9b'b'} + \text{etc.} + r^{(a-1)^2 b'b'}$$

Sed quum r^{aa} ſit radix propria aequationis $x^{bc} - 1 = 0$, erit ipſe factor prior productum ex

$$1 + \varrho^{bb} + \varrho^{4bb} + \varrho^{9bb} + \text{etc.} + \varrho^{(c-1)^2 bb}$$

$$1 + \varrho^{cc} + \varrho^{4cc} + \varrho^{9cc} + \text{etc.} + \varrho^{(b-1)^2 cc}$$

ſi ſtatuitur $r^{aa} = \varrho$. Hinc patet, W eſſe productum e factoribus tribus

$$1 + r^{bbcc} + r^{4bbcc} + r^{9bbcc} + \text{etc.} + r^{(a-1)^2 bbcc}$$

$$1 + r^{aacc} + r^{4aacc} + r^{9aacc} + \text{etc.} + r^{(b-1)^2 aacc}$$

$$1 + r^{aabb} + r^{4aabb} + r^{9aabb} + \text{etc.} + r^{(c-1)^2 aabb}$$

vbi r^{bbcc}, r^{aacc}, r^{aabb} erunt reſp. radices propriae aequationum $x^a - 1 = 0$, $x^b - 1 = 0$, $x^c - 1 = 0$.

27.

27.

Hinc facile concluditur generaliter, si n sit productum e factoribus quotcunque inter se primis a, b, c etc., W fieri productum e totidem factoribus, qui sint

$$1 + r^{\frac{nn}{aa}} + r^{\frac{4nn}{aa}} + r^{\frac{9nn}{aa}} + \text{etc.} + r^{\frac{(a-1)^2 nn}{aa}}$$

$$1 + r^{\frac{nn}{bb}} + r^{\frac{4nn}{bb}} + r^{\frac{9nn}{bb}} + \text{etc.} + r^{\frac{(b-1)^2 nn}{bb}}$$

$$1 + r^{\frac{nn}{cc}} + r^{\frac{4nn}{cc}} + r^{\frac{9nn}{cc}} + \text{etc.} + r^{\frac{(c-1)^2 nn}{cc}} \quad \text{etc.}$$

vbi $r^{\frac{nn}{aa}}$, $r^{\frac{nn}{bb}}$, $r^{\frac{nn}{cc}}$ etc. erunt radices propriae aequationum $x^a - 1 = 0$, $x^b - 1 = 0$, $x^c - 1 = 0$ etc.

28.

Ex his principiis transitus ad determinationem completam ipsius W pro valore quocunque ipsius n sponte iam obuius est. Decomponatur scilicet n in factores a, b, c etc. tales, qui sint vel numeri primi inaequales, vel potestates numerorum primorum inaequalium, statuatur $r^{\frac{nn}{aa}} = A$, $r^{\frac{nn}{bb}} = B$, $r^{\frac{nn}{cc}} = C$ etc., eruntque A, B, C etc. radices propriae aequationum $x^a - 1 = 0$, $x^b - 1 = 0$, $x^c - 1 = 0$ etc., atque W productum e factoribus

$$1 + A + A^4 + A^9 + \text{etc.} + A^{(a-1)^2}$$

$$1 + B + B^4 + B^9 + \text{etc.} + B^{(b-1)^2}$$

$$1 + C + C^4 + C^9 + \text{etc.} + C^{(c-1)^2} \quad \text{etc.}$$

Sed hi singuli factores per ea quae in artt. 20. 21. 23. docuimus, determinari poterunt, vnde etiam valor producti innotescet. Regulas pro determinandis illis factoribus hic in vnum obtutum collegisse haud inutile erit. Quum radix A fiat $= \frac{kn}{a} \cdot \frac{360°}{a}$, aggregatum $1 + A + A^4 + A^9 + \text{etc.} + A^{(a-1)^2}$, quod per L denotabimus,

notabimus, perinde per numerum $\frac{kn}{a}$ determinabitur, vt in disquisitione noſtra generali W per k. Duodecim iam caſus ſunt diſtinguendi.

I. Si a eſt numerus primus formae $4\mu+1$, puta $=p$, vel poteſtas talis numeri primi cum exponente impari, ſimulque $\frac{kn}{a}$ reſiduum quadraticum ipſius p, erit $L=+\sqrt{a}$.

II. Si manentibus reliquis, $\frac{kn}{a}$ eſt non-reſiduum quadraticum ipſius p, erit $L=-\sqrt{a}$.

III. Si a eſt numerus primus formae $4\mu+3$, puta $=p$, vel poteſtas talis numeri primi cum exponente impari, ſimulque $\frac{kn}{a}$ reſiduum quadraticum ipſius p, erit $L=+i\sqrt{a}$.

IV. Si manentibus reliquis vt in III, $\frac{kn}{a}$ eſt non-reſiduum quadraticum ipſius p, erit $L=-i\sqrt{a}$.

V. Si a eſt quadratum, altiorue poteſtas numeri primi (imparis) cum exponente pari, erit $L=+\sqrt{a}$.

VI. Si $a=2$, erit $L=0$.

VII. Si $a=4$, altiorue poteſtas binarii cum exponente pari, ſimulque $\frac{kn}{a}$ formae $4\mu+1$, erit $L=(1+i)\sqrt{a}$.

VIII. Si manentibus reliquis vt in VII, $\frac{kn}{a}$ eſt formae $4\mu+3$, erit $L=(1-i)\sqrt{a}$.

IX. Si $a=8$, altiorue poteſtas binarii cum exponente impari, ſimulque $\frac{kn}{a}$ formae $8\mu+1$, erit $L=(1+i)\sqrt{a}$.

X. Si

X. Si manentibus reliquis vt in IX, $\frac{kn}{a}$ est formae $8\mu + 3$, erit $L = (-1 + i)\sqrt{a}$.

XI. Si manentibus reliquis $\frac{kn}{a}$ est formae $8\mu + 5$, erit $L = (-1 - i)\sqrt{a}$.

XII. Si manentibus reliquis $\frac{kn}{a}$ est formae $8\mu + 7$, erit $L = (1 - i)\sqrt{a}$.

29.

Sit exempli caussa $n = 2520 = 8.9.5.7$, atque $k = 13$. Hic erit
pro $a = 8$, per casum XII, $L = (1 - i)\sqrt{8}$
pro factore 9, per casum V, summa respondens erit $= \sqrt{9}$
pro factore 5, per casum II, summa respondens erit $= -\sqrt{5}$
pro factore 7, per casum III, summa respondens erit $= + i\sqrt{5}$
Hinc fit $W = (1 - i).(-i).\sqrt{2520} = (-1 - i)\sqrt{2520}$.

Sit pro eodem valore ipsius n, $k = 1$: tunc respondebit
factori 8 summa $(-1 + i)\sqrt{8}$
factori 9 summa $\sqrt{9}$
factori 5 summa $\sqrt{5}$
factori 7 summa $-i\sqrt{7}$.
Hinc conflatur productum $W = (1 + i)\sqrt{2520}$.

30.

Methodus alia, summam W generaliter determinandi, petitur ex iis, quae in artt. 22. 24. exposita sunt. Statuamus $\cos\omega + i \sin\omega = \varrho$, atque $\varrho^{\frac{nn}{aa}} = \alpha$, $\varrho^{\frac{nn}{bb}} = \beta$, $\varrho^{\frac{nn}{cc}} = \gamma$ etc., ita vt habeatur $r = \varrho^k$, $A = \alpha^k$, $B = \beta^k$, $C = \gamma^k$ etc. Tunc erit
$1 + \varrho + \varrho^4 + \varrho^9 + \text{etc.} + \varrho^{(n-1)^2}$

productum

productum e factoribus

$$1+\alpha+\alpha^4+\alpha^9+\text{etc.}+\alpha^{(a-1)^2}$$
$$1+\beta+\beta^4+\beta^9+\text{etc}+\beta^{(b-1)^2}$$
$$1+\gamma+\gamma^4+\gamma^9+\text{etc.}+\gamma^{(c-1)^2}\quad\text{etc.}$$

adeoque W productum e factoribus

$$w=1+\varrho+\varrho^4+\varrho^9+\text{etc.}+\varrho^{(n-1)^2}$$
$$\mathfrak{A}=\frac{1+A+A^4+A^9+\text{etc.}+A^{(a-1)^2}}{1+\alpha+\alpha^4+\alpha^9+\text{etc.}+\alpha^{(a-1)^2}}$$
$$\mathfrak{B}=\frac{1+B+B^4+B^9+\text{etc.}+B^{(b-1)^2}}{1+\beta+\beta^4+\beta^9+\text{etc.}+\beta^{(b-1)^2}}$$
$$\mathfrak{C}=\frac{1+C+C^4+C^9+\text{etc.}+C^{(c-1)^2}}{1+\gamma+\gamma^4+\gamma^9+\text{etc.}+\gamma^{(c-1)^2}}\quad\text{etc.}$$

Iam factor primus w determinatus est per disquisitiones supra traditas (art. 19.); factores reliqui vero $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. prodeunt per formulas artt. 22. 24., quas vt omnia iuncta habeantur hic denuo colligimus *). Duodecim casus hic sunt distinguendi, scilicet

I. Si a est numerus primus (impar) $=p$, vel talis numeri potestas cum exponente impari, atque k residuum quadraticum ipsius p, erit factor respondens $\mathfrak{A}=+1$.

II. Si manentibus reliquis k est non-residuum quadraticum ipsius p, erit $\mathfrak{A}=-1$.

III. Si a est quadratum numeri primi imparis, altiorue eius potestas cum exponente pari, erit $\mathfrak{A}=+1$.

IV. Si a est $=4$, aut altior binarii potestas cum exponente pari, simulque k formae $4\mu+1$, erit $\mathfrak{A}=+1$.

V. Si,

*) Manifesto, quae illic erant k et h, hic erunt $\frac{n}{a}$ et k respectu factoris secundi, $\frac{n}{b}$ et k respectu factoris tertii etc.

V. Si, manentibus reliquis vt in IV, k eſt formae $4\mu+3$, atque $\frac{n}{a}$ formae $4\mu+1$, erit $\mathfrak{A}=-i$.

VI. Si, manentibus reliquis vt in IV, k eſt formae $4\mu+3$, atque $\frac{n}{a}$ formae $4\mu+3$, erit $\mathfrak{A}=+i$.

VII. Si a eſt $=8$, aut altior binarii poteſtas cum exponente impari, atque k formae $8\mu+1$, erit $\mathfrak{A}=+1$.

VIII. Si, manentibus reliquis vt in VII, k eſt formae $8\mu+5$, erit $\mathfrak{A}=-1$.

IX. Si, manentibus reliquis vt in VII, k eſt formae $8\mu+3$, atque $\frac{n}{a}$ formae $4\mu+1$, erit $\mathfrak{A}=+i$.

X. Si, manentibus reliquis vt in VII, k eſt formae $8\mu+3$, atque $\frac{n}{a}$ formae $4\mu+3$, erit $\mathfrak{A}=-i$.

XI. Si, manentibus reliquis vt in VII, k eſt formae $8\mu+7$, atque $\frac{n}{a}$ formae $4\mu+1$, erit $\mathfrak{A}=-i$.

XII. Si, manentibus reliquis vt in VII, k eſt formae $8\mu+7$, atque $\frac{n}{a}$ formae $4\mu+3$, erit $\mathfrak{A}=+i$.

Caſum eum, vbi $a=2$, praeterimus; hic quidem $\mathfrak{A}$ foret $=\frac{0}{0}$ ſiue indeterminatus, ſed tunc ſemper $W=0$.

Factores reliqui $\mathfrak{B}$, $\mathfrak{C}$ etc. perinde pendent a b, c etc., vt $\mathfrak{A}$ ab a, quatenus in illorum determinationem ingrediuntur.

31.

31.

Secundum hanc methodum alteram exemplum primum art. 29. ita se habet:

Factor w fit $= (1 + i)\sqrt{2520}$.

Pro $a = 8$ factor respondens $\mathfrak{A}$ fit, per casum VIII, $= -1$.

Factori ipsius n secundo 9 respondet factor $+ 1$ (per casum III.).

Factori 5 respondet factor $- 1$ (per casum II.).

Factori 7 respondet factor $- 1$ (per casum II.).

Hinc conflatur productum $W = (-1 - i)\sqrt{2520}$, vt in art. 29.

32.

Quum valor ipsius W per methodos *duas* determinari possit, quarum altera relationibus numerorum $\frac{nk}{a}$, $\frac{nk}{b}$, $\frac{nk}{c}$ etc. ad numeros a, b, c etc. innititur, altera vero a relationibus ipsius k ad numeros a, b, c etc. pendet, inter omnes has relationes nexus quidam conditionalis intercedere debet, ita vt quaeuis e reliquis determinabilis esse debeat. Supponamus, omnes numeros a, b, c etc. esse numeros primos impares, atque k accipi $= 1$; distribuanturque factores a, b, c etc. in duas classes, quarum altera contineat eos, qui sunt formae $4\mu + 1$, et qui denotentur per p, p', p'' etc., altera vero constet ex iis, qui sunt formae $4\mu + 3$, et qui exprimantur per q, q', q'' etc.: multitudinem posteriorum designabimus per m. His ita factis, obseruamus primo, n fieri formae $4\mu + 1$, si m fuerit par (quorsum etiam referri debet casus is, vbi factores classis alterius omnino desunt, siue vbi $m = 0$), contra n fieri formae $4\mu + 3$, si m fuerit impar. Iam determinatio ipsius W per methodum primam ita perficitur. Pendeant numeri P, P', P'' etc., Q, Q', Q'' etc. ita a relationibus numerorum $\frac{n}{p}$, $\frac{n}{p'}$, $\frac{n}{p''}$ etc., $\frac{n}{q}$, $\frac{n}{q'}$, $\frac{n}{q''}$ etc. ad numeros p, p', p'' etc., q, q', q'' etc. resp., vt statuatur

$P = +1$,

$P = +1$, si $\frac{n}{p}$ est residuum quadraticum ipsius p

$P = -1$, si $\frac{n}{p}$ est non-residuum quadraticum ipsius p

et perinde de reliquis. Tunc erit W productum e factoribus $P\sqrt{p}$, $P'\sqrt{p'}$, $P''\sqrt{p''}$ etc., $iQ\sqrt{q}$, $iQ'\sqrt{q'}$, $iQ''\sqrt{q''}$ etc., adeoque

$W = PP'P'' \ldots QQ'Q'' \ldots i^m\sqrt{n}$

Per methodum secundam, aut potius statim per praecepta art. 19., erit

$W = +\sqrt{n}$, si n est formae $4\mu+1$, vel quod eodem redit, si m est par,

$W = +i\sqrt{n}$, si n est formae $4\mu+3$, vel si m est impar.

Vtrumque casum simul complecti licet per formulam sequentem:

$W = i^{mm}\sqrt{n}$.

Hinc itaque colligitur

$PP'P'' \ldots QQ'Q'' \ldots = i^{mm-m}$

Sed i^{mm-m} fit $=1$, quoties m est formae 4μ vel $4\mu+1$, atque $=-1$, quoties m est formae $4\mu+2$ vel $4\mu+3$, vnde deducimus sequens elegantissimum

Theorema. Denotantibus a, b, c etc. numeros primos impares positiuos inaequales, quorum productum statuitur $=n$, et inter quos m sint formae $4\mu+3$, reliqui formae $4\mu+1$: multitudo eorum ex his numeris a, b, c etc., quorum non-residua resp. sunt $\frac{n}{a}$, $\frac{n}{b}$, $\frac{n}{c}$ etc., par erit, quoties m est formae 4μ vel $4\mu+1$, impar vero, quoties m est formae $4\mu+2$ vel $4\mu+3$.

Ita e.g. statuendo $a=3$, $b=5$, $c=7$, $d=11$, habemus tres numeros formae $4\mu+3$, puta 3, 7 et 11; est autem $5.7.11\,R3$; $3.7.11\,R5$; $3.5.11\,R7$; $3.5.7\,N11$, siue vnicus $\frac{n}{d}$ est non-residuum ipsius d.

33.

33.

Celeberrimum *theorema fundamentale* circa residua quadratica nihil aliud est, nisi casus specialis theorematis modo euoluti. Limitando scilicet multitudinem numerorum a, b, c etc. ad *duos*, patet, si vnus tantum ex ipsis, vel neuter, sit formae $4\mu+3$, fieri debere vel simul aRb, bRa, vel simul aNb, bNa; contra si vterque est formae $4\mu+3$, vnus ex ipsis alterius non-residuum esse debebit, atque hic illius residuum. En itaque demonstrationem *quartam* huius grauissimi theorematis, cuius demonstrationem primam et secundam in Disquisitionibus Arithmeticis, tertiam nuper in commentatione peculiari tradidimus (*Commentt. T. XVI p. 69*): duas alias principiis rursus omnino diuersis innitentes in posterum exponemus. Summopere sane est mirandum, quod hocce venustissimum theorema, quod primo omnes conatus tam pertinaciter eluserat, tot postea viis toto coelo inter se distantibus adiri potuerit.

34.

Etiam theoremata reliqua, quae quasi supplementum ad theorema fundamentale efficiunt, scilicet per quae dignoscuntur numeri primi, quorum residua vel non-residua sunt -1, $+2$ et -2, ex iisdem principiis deriuari possunt. Incipiemus a residuo $+2$.

Statuendo $n=8a$, ita vt a sit numerus primus, atque $k=1$, per methodum art 28. W erit productum e duobus factoribus, quorum alter erit $+\sqrt{a}$, vel $+i\sqrt{a}$, si 8, vel quod idem est 2, est residuum quadraticum ipsius a; contra $\sqrt{a}$ vel $-i\sqrt{a}$, si 2 est non-residuum ipsius a. Factor secundus autem est

$(1+i)\sqrt{8}$, si a est formae $8\mu+1$

$(-1+i)\sqrt{8}$, si a est formae $8\mu+3$

$(-1-i)\sqrt{8}$, si a est formae $8\mu+5$

$(1-i)\sqrt{8}$, si a est formae $8\mu+7$

Sed

Sed per art. 18. semper erit $W = (1+i)\sqrt{n}$; diuidendo hunc valorem per quatuor valores factoris secundi, patet, factorem primum fieri debere

$+\sqrt{a}$, si a est formae $8\mu+1$

$-i\sqrt{a}$, si a est formae $8\mu+3$

$-\sqrt{a}$, si a est formae $8\mu+5$

$i\sqrt{a}$, si a est formae $8\mu+7$.

Hinc sponte sequitur, in casu primo et quarto 2 esse debere residuum ipsius a, in casu secundo et tertio autem non-residuum.

35.

Numeri primi, quorum residuum vel non-residuum est -1, facile dignoscuntur adiumento theorematis sequentis, quod etiam per se ipsum satis memorabile est.

Theorema. Productum e duobus factoribus

$$W' = 1 + r^{-1} + r^{-4} + \text{etc.} + r^{-(n-1)^2}$$

$$W = 1 + r + r^4 + \text{etc.} + r^{(n-1)^2}$$

est $= n$, si n est impar; vel $= 0$, si n est impariter par; vel $= 2n$, si n est pariter par.

Demonstr. Quum manifesto fiat

$$W = r + r^4 + r^9 + \text{etc.} + r^{nn}$$
$$= r^4 + r^9 + \text{etc.} + r^{(n+1)^2}$$
$$= r^9 + \text{etc.} + r^{(n+2)^2} \text{ etc.}$$

productum $W'W'$ ita quoque exhiberi poterit

$$1 + r + r^4 + r^9 + \text{etc.} + r^{(n-1)^2}$$
$$+ r^{-1}(r + r^4 + r^9 + r^{16} + \text{etc.} + r^{nn})$$
$$+ r^{-4}(r^4 + r^9 + r^{16} + r^{25} + \text{etc.} + r^{(n+1)^2})$$
$$+ r^{-9}(r^9 + r^{16} + r^{25} + r^{36} + \text{etc.} + r^{(n+2)^2})$$
$$\text{etc.}$$
$$+ r^{-(n-1)^2}(r^{(n-1)^2} + r^{nn} + r^{(n+1)^2} + r^{(n+2)^2} + \text{etc.} + r^{(2n-2)^2})$$

quod

quod aggregatum verticaliter fummatum producit

$$n$$
$$+ r(1 + rr + r^4 + r^6 + \text{etc.} + r^{2n-2})$$
$$+ r^4(1 + r^4 + r^8 + r^{12} + \text{etc.} + r^{4n-4})$$
$$+ r^9(1 + r^6 + r^{12} + r^{18} + \text{etc.} + r^{6n-6})$$
$$+ \text{etc.}$$
$$+ r^{(n-1)^2}(1 + r^{2n-2} + r^{4n-4} + r^{6n-6} + \text{etc.} + r^{2(n-1)^2})$$

Iam fi n impar eft, fingulae partes huius aggregati, praeter primam n, erunt $= 0$; fecunda enim manifefto fit $\frac{r(1-r^{2n})}{1-rr}$, tertia $\frac{r^4(1-r^{4n})}{1-r^4}$ etc. Quoties vero n impar eft, excipere infuper oportebit partem

$$r^{\frac{1}{4}nn}(1 + r^n + r^{2n} + r^{3n} + \text{etc.} + r^{nn-n})$$

quae fit $= nr^{\frac{1}{4}nn}$. In cafu priori itaque fit $WW' = n$, in pofteriori autem $= n + nr^{\frac{1}{4}nn}$; fed $r^{\frac{1}{4}nn}$ fit $= +1$, fi n eft pariter par, tunc itaque prodit $WW' = 2n$; contra fit $r^{\frac{1}{4}nn} = -1$, fi n eft impariter par, vbi itaque euadit $WW' = 0$. *Q. E. D.*

36.

Iam per art. 22. conftat, fi n fit numerus primus impar, $\frac{W'}{W}$ fieri $= +1$ vel $= -1$, prout -1 fuerit refiduum vel nonrefiduum ipfius n. Hinc in cafu priori effe debebit $W^2 = +n$, in pofteriori $W^2 = -n$; quamobrem per art. 13. concludimus, cafum priorem tunc tantum locum habere poffe, quando n fit formae

formae $4\mu + 1$, casumque posteriorem, quando n sit formae $4\mu + 3$.

Denique e combinatione conditionum pro residuis $+2$ et -1 inuentarum sponte sequitur, -2 esse residuum cuiusuis numeri primi formae $8\mu + 1$ vel $8\mu + 3$, atque non-residuum cuiusuis numeri primi formae $8\mu + 5$ vel $8\mu + 7$.